The Long-Range Ising Model in 1D at Inverse-Square Decay

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Master's Defense — September 2, 2026

Table of Contents

The Classical Ising Model

Definition (Ising Model)

Configuration $\sigma = (\sigma_i)_{i\in\Lambda} \in \{-1,+1\}^\Lambda$ on $\Lambda \subset \mathbb{Z}^d$ with $J_{ij} \geq 0$:

$$H_\Lambda(\sigma) = -\sum_{\{i,j\}\subset\Lambda} J_{ij}\sigma_i\sigma_j, \qquad \mu_{\Lambda,\beta}(\sigma) = \frac{1}{Z_{\Lambda,\beta}} e^{-\beta H_\Lambda(\sigma)}.$$

Spontaneous Magnetization & Critical Point

$$m^+(\beta) := \lim_{\Lambda\uparrow\mathbb{Z}^d} \langle\sigma_0\rangle^+_{\Lambda,\beta}, \qquad \beta_c := \inf\{\beta > 0 : m^+(\beta) > 0\}.$$

Theorem (Ising 1925: 1D Nearest-Neighbor)

$$\beta_c = \infty \qquad \Longleftrightarrow \qquad m^+(\beta) = 0 \quad (\forall\beta > 0).$$

Long-Range Ising Model in 1D at Inverse-Square Decay

Theorem (1D Long-Range Ising Model at Inverse-Square Decay)

Fröhlich – Spencer 1982, Aizenman – Chayes – Chayes – Newman 1988

For the 1D Long-Range Ising Model with $J(r) \asymp r^{-2}$ $(\alpha = 2)$:

$$\begin{cases} \text{Phase Transition [FS82]} : & \exists\, \beta_c > 0 \text{ s.t. } m^*(\beta) > 0 \quad (\forall \beta > \beta_c), \\ \text{Discontinuity [ACCN88]} : & m^*(\beta_c) = \lim\limits_{\beta \searrow \beta_c} m^*(\beta) > 0. \end{cases}$$

Specificity of the Problem at Inverse-Square Decay

Droplet Energy Cost (Thouless 1969, Landau-Lifshitz 1980)

For an interval of flipped spins $I_\ell = [\![1, \ell]\!] \subset \mathbb{Z}$ of length ℓ:

$$\Delta E(\ell) = 2 \sum_{i \in I_\ell} \sum_{j \notin I_\ell} J(|i-j|).$$

Energy Scaling ($J(r) \asymp r^{-\alpha}$)

$$\Delta E(\ell) \asymp \begin{cases} \ell^{2-\alpha}, & 1 < \alpha < 2, \\ \log \ell, & \alpha = 2, \\ 1, & \alpha > 2. \end{cases}$$

Duminil-Copin – Garban – Tassion (2025)

1D Long-Range Ising Model Revisited

H. Duminil-Copin, C. Garban, V. Tassion (2025)

ACCN (1988) strategy

Long-Range Bernoulli Percolation → **FK Percolation ($q = 2$)** → **1D Long-Range Ising Model**

The Long-Range Bernoulli Percolation

Model on the complete graph over $\mathbb{Z}$

Edges are independent and

$$p_{i,j}(\beta,\lambda) = \begin{cases} 1 - \exp\left(-\frac{\beta}{|i-j|^2}\right), & |i-j| \geq 2, \\ 1 - \exp(-\lambda), & |i-j| = 1. \end{cases}$$

$$\theta(\beta,\lambda) = \mathbb{P}_{\beta,\lambda}(0 \leftrightarrow \infty).$$

Theorem I (DCGT 2025)

(i) $\beta > 1 \Rightarrow \theta(\beta,\lambda) > 0$ for λ large, (ii) $\theta > 0 \Rightarrow \beta\theta^2 \geq 1$.

Proof of Theorem I(i)

- **Blocks:** $B_K^i = [K(i-1), K(i+1))$, $\qquad B_K = B_K^0$
- **θ-good Block:** It is θ-good if it contains a cluster with at least $2\theta K$ vertices
- **θ-bad probability:**
  $p_{\beta,\lambda}(K,\theta) := \mathbb{P}_{\beta,\lambda}[B_K \text{ is } \theta\text{-bad}] = 1 - \mathbb{P}_{\beta,\lambda}[B_K \text{ is } \theta\text{-good}]$.

Lemma (DCGT Lemma 2)

Choose $\theta_\infty \in (3/4, 1)$ such that $\beta\theta_\infty^2 > 1$. There exists a constant $C_0 > 0$ such that for all $\theta \geq \theta_\infty$, $C \geq C_0$ and $\theta' := \theta - \frac{C_0}{C}$:

$$p_{\beta,\lambda}(CK, \theta') \leq \frac{1}{100} p_{\beta,\lambda}(K,\theta) + 2C^2 p_{\beta,\lambda}(K,\theta)^2.$$

Proof of Lemma 2

- **Event decomposition:** For $I_C = \{-C+1, \ldots, C-1\}$:

$$\{B_{CK} \text{ is } \theta'\text{-bad}\} \subset \bigcup_{i \in I_C} F_i \cup D, \qquad \text{where } F_i = E_i \cap \{B_{CK} \text{ is } \theta'\text{-bad}\}.$$

  - E_i: B_K^i is θ-bad, and all bad blocks are in $B_K^{i-1} \cup B_K^i \cup B_K^{i+1}$ (defect zone).
  - D: contains two vertex-disjoint θ-bad K-blocks.

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- **Union bound:**

$$p_{\beta,\lambda}(CK, \theta') \leq \sum_{i \in I_C} \mathbb{P}(F_i) + \underbrace{\mathbb{P}(D)}_{\substack{\text{(Bernoulli independence)} \\ \leq \binom{2C-1}{2} p_{\beta,\lambda}(K,\theta)^2 \leq 2C^2 p_{\beta,\lambda}(K,\theta)^2}} .$$

Proof of Lemma 2: Single Defect Case

good run	$B_K^{i-1} \cup B_K^i \cup B_K^{i+1}$	good run
C_i^-	**defect zone**	C_i^+

$$\boxed{\text{On } E_i: \ \{B_{CK} \text{ is } \theta'\text{-bad}\} \subset \{\text{all edges between } C_i^-, C_i^+ \text{ are closed}\}}$$

$$\Downarrow$$

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$$\boxed{F_i \subset E_i \cap \{\text{all edges between } C_i^-, C_i^+ \text{ are closed}\}}$$

$\Downarrow$ **Bernoulli independence**

$$\mathbb{P}(F_i) \leq \mathbb{P}(E_i) \exp\left(-\beta \sum_{x \in C_i^-} \sum_{y \in C_i^+} \frac{1}{|x-y|^2} \right).$$

Proof of Lemma 2: One-Defect Estimate

On E_i, ordering vertices in C_i^-, C_i^+ gives $\theta(y_b - x_a) \leq 6K + a + b \;\; (d_i = C - |i|)$:

$$\boxed{\sum_{x \in C_i^-} \sum_{y \in C_i^+} \frac{1}{|x-y|^2} \;\geq\; \theta^2 \sum_{a,b \leq \frac{Kd_i}{2}} \frac{1}{(6K+a+b)^2} \;\geq\; \theta^2 \log\left(\frac{d_i}{48}\right)}$$

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$$\Downarrow$$

$$\boxed{\mathbb{P}(F_i) \ \le\ \mathbb{P}(E_i)\left(\frac{48}{d_i}\right)^{\beta\theta^2} \ \le\ p_{\beta,\lambda}(K,\theta)\left(\frac{48}{d_i}\right)^{\beta\theta^2}}$$

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$\Downarrow$ **since** $\beta\theta^2 > 1$ **and** C **large**

$$\sum_{i \in I_C} \mathbb{P}(F_i) \;\leq\; \frac{1}{100}\, p_{\beta,\lambda}(K,\theta).$$

From Lemma 2 to Theorem I(i)

Lemma 2

$$p(CK,\theta') \leq \frac{1}{100} p(K,\theta) + 2C^2 p(K,\theta)^2$$

$\Downarrow$ **choose scales and densities:**

$$K_{n+1} = C_{n+1} K_n,\ \theta_{n+1} = \theta_n - \tfrac{C_0}{C_{n+1}} \implies \theta_n \geq \theta_\infty > \tfrac{3}{4}$$

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Induction with $u_n := p(K_n, \theta_n)$: $\quad u_{n+1} \leq \dfrac{1}{100} u_n + 2C_{n+1}^2 u_n^2.$

$\Downarrow$ **Initialisation by large λ:** u_1 small $\implies u_n \leq \frac{\delta}{C_n^2}, \ \mathbb{P}(B_{K_n} \text{ is } \theta_n\text{-good}) \geq \frac{1}{2}$

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Cluster overlap:

$$\theta_n > \tfrac{3}{4} \implies \mathbb{P}\big(0 \text{ belongs to a cluster of size } \geq \tfrac{3}{2}K_n\big) \geq \tfrac{3}{8}.$$

$\Downarrow$ **Letting** $n \to \infty$

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Theorem I(i)

$$\beta > 1 \implies \theta(\beta, \lambda) > 0 \quad (\text{for } \lambda \text{ sufficiently large})$$

Proof of Theorem I(ii): Failed Crossings

Definition (K-crossed Block & Observable $p(K)$)

For $B^i_{3K} = [3K(i-1), 3K(i+1))$:

- **K-crossed:** $\exists\, x < 3Ki - 3K,\ y \geq 3Ki + 3K$ connected by an open path using only edges of length $|e| \leq K$.
- **Failure probability:** $p(K) := \mathbb{P}_{\beta,\lambda}\big(B_{3K} \text{ is not } K\text{-crossed}\big)$.

Proof by contradiction: Assume $\theta > 0,\ \beta\theta^2 < 1$. Choose $\theta < \vartheta < \frac{1}{\sqrt{\beta}}$ $(\beta\vartheta^2 < 1)$

$\Downarrow$ **Contradiction Strategy**

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$\Downarrow$ **Contradiction Strategy**

Show persistent obstruction: $\mathbf{p(K_n) \geq c > 0}$ for infinitely many $K_n \to \infty$

Candidate Blocks and Bridges

Inside B_{CK}, consider separated candidate blocks
$I_C^{\mathrm{cand}} := \{i \in 3\mathbb{Z} : B_{3K}^i \subset B_{CK}\}$:

- Edge lengths: $|e| \le K$ (**short**), $\quad K < |e| \le CK$ (**long**).
- **Bridge:** A long open edge $e = \{x, y\}$ whose endpoints have open R-arms off e.
- **Unbridged block:** B_{3K}^i bypassed by no long edge bridge.

Unbridged Candidate Blocks U

$$U := \left\{ B_{3K}^i : i \in I_C^{\mathrm{cand}},\ B_{3K}^i \text{ is unbridged} \right\}.$$

$$\boxed{B_{3CK} \text{ is } CK\text{-crossed} \implies \text{every block } B \in U \text{ is } K\text{-crossed}}$$

Failed-Crossing Renormalisation

Conditioning on long edges $\mathcal{F}$ (disjoint short-edge families):

$$\mathbb{P}(\text{every } B \in U \text{ is crossed} \mid \mathcal{F}) = (1 - p(K))^{|U|}$$

$$\Downarrow$$

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$$p(CK) \geq 1 - \mathbb{E}\left[(1 - p(K))^{|U|}\right] \;\geq\; e^{-1}\, \mathbb{E}[|U|] \min\{p(K),\, C^{-1}\}$$

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$\Downarrow$ **Inverse-square tail:**

$$\sum_{K \leq r \leq CK} \frac{\beta\vartheta^2}{r^2} \sim \beta\vartheta^2 \log C \implies \mathbb{E}[|\mathbf{U}|] \gtrsim \mathbf{C}^{1-\beta\vartheta^2}$$

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Lemma (DCGT: Failed-Crossing Renormalisation)

$$p(CK) \geq C^{1-\beta\vartheta^2}\min\{p(K),\, C^{-1}\} \implies \mathbf{p(K_n)} \geq \eta > \mathbf{0} \quad (\forall \mathbf{n} \geq \mathbf{n_0}).$$

Annulus Obstruction & Contradiction

Annulus-Crossing Event $A(K) := \{B_{3K} \longleftrightarrow B_{9K}^c\}$

Two failed K-crossings + no long bypass across annulus $\implies A(K)^c$.

- **From failed-crossing lower bound ($p(K_n) \geq \eta > 0$):**
$$1 - \mathbb{P}(A(K_n)) \geq c_\beta\, p(K_n)^2 \implies \mathbb{P}(\mathbf{A(K_n)}) \leq \mathbf{1} - \mathbf{c} < \mathbf{1}.$$

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- **From percolation assumption ($\theta > 0$) and ergodicity:**
$$\mathbb{P}(\mathbf{A}(\mathbf{K_n})) \xrightarrow[\mathbf{n}\to\infty]{} \mathbf{1}.$$

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$\Downarrow$ **Contradiction!**

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$\Downarrow$ **Contradiction!**

Theorem I(ii) (Density Gap, DCGT 2025)

$$\theta(\beta, \lambda) > 0 \implies \boxed{\beta\, \theta(\beta, \lambda)^2 \geq 1.}$$

FK Percolation & Theorem II

FK Percolation / Random-Cluster Model ($q \geq 1$)

For an edge $e = \{x, y\}$, conditional on $\omega_{-e} := \omega|_{E\setminus\{e\}}$:

$$\mathbb{P}_{p_e,q}\big(e \text{ is open} \,\big|\, \omega_{-e}\big) = \begin{cases} p_e, & \text{if } x \longleftrightarrow y \text{ in } \omega_{-e}, \\ \dfrac{p_e}{p_e + (1-p_e)q}, & \text{if } x \not\longleftrightarrow y \text{ in } \omega_{-e}. \end{cases}$$

Theorem II (Density Gap for FK Percolation, DCGT 2025)

Let $\theta(q, \beta, \lambda) := \mathbb{P}^{\text{wired}}_{q,\beta,\lambda}(0 \leftrightarrow \infty)$ ($q \geq 1$):

- (i) If $\beta > 1$, then $\theta(q, \beta, \lambda) > 0$ for all sufficiently large finite λ.
- (ii) For every $\beta, \lambda > 0$,

$$\theta(q, \beta, \lambda) > 0 \implies \boxed{\beta\, \theta(q, \beta, \lambda)^2 \geq 1.}$$

Edwards–Sokal Coupling & Discontinuity

Edwards–Sokal Coupling (1988) at $q = 2$

For H_{Ising} with $J_{i,j} = \frac{1}{|i-j|^2}$, ES gives $\beta = 2\beta_I$ and $\lambda = 2J_1\beta_I$:

$$\langle \sigma_0 \rangle^+_{\Lambda,\beta_I} = \phi^{\mathrm{wired}}_{\Lambda,p,2}(0 \leftrightarrow \partial\Lambda) \xrightarrow[\Lambda\uparrow\mathbb{Z}]{} m^+(\beta_I) = \theta(2, 2\beta_I, 2J_1\beta_I).$$

Corollary (Phase Transition & Discontinuity)

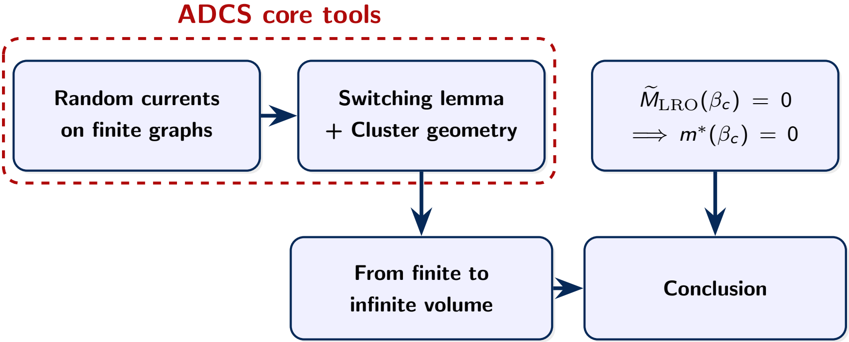
By Theorem II with $q = 2$:

1. **Phase transition (from (i)):** $\beta_{I,c} := \inf\{\beta_I > 0 : m^+(\beta_I) > 0\} > 0.$
2. **Discontinuity (from (ii)):** $m^+(\beta_I) > 0 \implies 2\beta_I\, m^+(\beta_I)^2 \geq 1$, which yields:

$$\boxed{m^+(\beta_{I,c}) = \lim_{\beta_I \downarrow \beta_{I,c}} m^+(\beta_I) \geq \frac{1}{\sqrt{2\beta_{I,c}}} > 0.}$$

Random Currents and Continuity of Ising Model's Spontaneous Magnetization

Aizenman – Duminil-Copin – Sidoravicius (2015)



Random Currents for Ising Model

Partition function on finite graph $\Lambda = (\Lambda, E_\Lambda)$:

$$Z_\Lambda = \sum_{\sigma \in \{\pm 1\}^\Lambda} \prod_{\{i,j\}} \exp(\beta_I J_{ij} \sigma_i \sigma_j)$$

$\Downarrow$ **Taylor expansion:** $e^{\beta_I J_{ij} \sigma_i \sigma_j} = \sum_{n_{ij} \geq 0} \frac{(\beta_I J_{ij})^{n_{ij}}}{n_{ij}!} (\sigma_i \sigma_j)^{n_{ij}}$

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$$\boxed{Z_\Lambda = \sum_{\mathbf{n} \in \mathbb{N}_0^{E_\Lambda}} \prod_{e \in E_\Lambda} \frac{(\beta_I J_e)^{n_e}}{n_e!} \prod_{i \in \Lambda} \left(\sum_{\sigma_i = \pm 1} \sigma_i^{d_{\mathbf{n}}(i)} \right)}$$

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$\Downarrow$ **since** $\sum_{\sigma_i = \pm 1} \sigma_i^k = \begin{cases} 2, & k \text{ is even,} \\ 0, & k \text{ is odd.} \end{cases}$

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$\Downarrow$ **since** $\sum_{\sigma_i = \pm 1} \sigma_i^k = \begin{cases} 2, & k \text{ is even}, \\ 0, & k \text{ is odd}. \end{cases}$

$$Z_\Lambda = 2^{|\Lambda|} \sum_{\partial \mathbf{n} = \varnothing} \prod_{e \in E_\Lambda} \frac{(\beta_I J_e)^{n_e}}{n_e!} =: \ 2^{|\Lambda|} Z_\Lambda(\varnothing)$$

Currents & sources: $w(\mathbf{n}) := \prod_{e \in E_\Lambda} \frac{(\beta_I J_e)^{n_e}}{n_e!}$, $\quad \partial \mathbf{n} := \{ i \in \Lambda : d_{\mathbf{n}}(i) \text{ is odd} \}$.

Switching Lemma for Random Currents

Current Representation: For any even set $A \subset \Lambda$,

$$\left\langle \prod_{i \in A} \sigma_i \right\rangle_{\Lambda, \beta_I} = \frac{2^{|\Lambda|} Z_\Lambda(A)}{2^{|\Lambda|} Z_\Lambda(\varnothing)} = \frac{Z_\Lambda(A)}{Z_\Lambda(\varnothing)}, \qquad Z_\Lambda(A) := \sum_{\partial \mathbf{n} = A} w(\mathbf{n})$$

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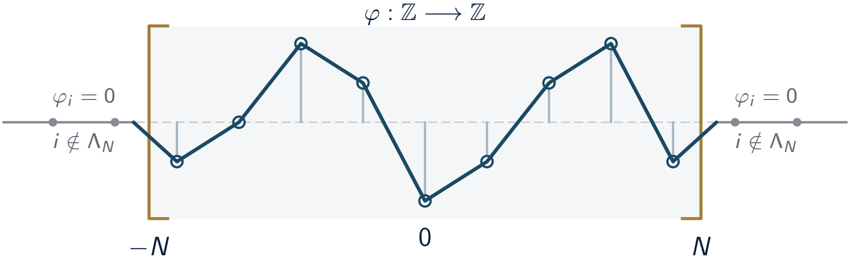
Discrete Gaussian Chain (DGC)

Definition (Discrete Gaussian Chain / 1D Integer-Valued GFF)

$$\mu^{\mathrm{DGC}}_{N,\beta,\alpha}(\varphi) = \frac{1}{Z_{N,\beta,\alpha}} \exp\left(-\frac{\beta}{2} \sum_{\substack{i,j\in\mathbb{Z},\, i\neq j \\ \{i,j\}\cap\Lambda_N\neq\emptyset}} J_\alpha(|i-j|)(\varphi_i - \varphi_j)^2 \right)$$



Theorem: (De-)Localisation for the DGC

Theorem ((De-)Localisation for the DGC)

Kjaer - Hilhorst 1984, Fröhlich – Zegarlinski 1991

$\forall \beta \in \mathbb{R}^+, \forall N \in \mathbb{N}^*,$

$$\begin{cases} \text{Delocalization :} & \mathrm{Var}^{\mathrm{DGC}}_{N,\beta,2}(\varphi_0) \asymp \log N, \quad \beta \ll 1 \quad [\mathrm{KH84}], \\ \text{Localization :} & \sup_N \mathrm{Var}^{\mathrm{DGC}}_{N,\beta,2}(\varphi_0) < \infty, \quad \beta \gg 1^* \quad [\mathrm{FZ91}]. \end{cases}$$

* This does not imply that the critical point is 1.

Open Problem 4 (C. Garban)

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Is the (de)localization phase transition discontinuous at the critical point β_c?*

* The numerical result was proven by Slurink and Hilhorst in 1983.

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1D Long-Range Ising Model

at inverse-square decay ($\alpha = 2$)

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Current-Type Expansion for the DGC

Rewrite DGC Hamiltonian:

$$e^{-\beta_D H_N^{\mathrm{DGC}}(\phi)} = \prod_{i \in \Lambda_N} e^{-\beta_D a \phi_i^2} \prod_{\{i,j\} \in E_N} e^{2\beta_D J_{ij} \phi_i \phi_j}, \quad a := \sum_{j \neq i} J_{ij}$$

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$\Downarrow$ **Define one-site moments:** $m_{\beta_D}(k) := \sum_{m\in\mathbb{Z}} m^k e^{-\beta_D a m^2}$

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Currents & degrees:

$\mathbf{n} = (n_e)_{e\in E_N} \in \mathbb{N}_0^{E_N}, \quad d_{\mathbf{n}}(i) := \sum_{e\ni i} n_e, \quad m_{\beta_D}(k) = 0$ for odd k.

DGC Variance in Current Representation

$$\boxed{\textbf{Target observable in Garban's problem:} \quad \mathsf{Var}^{\mathrm{DGC}}_{N,\beta_D,2}(\phi_0) = \mathbb{E}[\phi_0^2]}$$

$$\Downarrow$$

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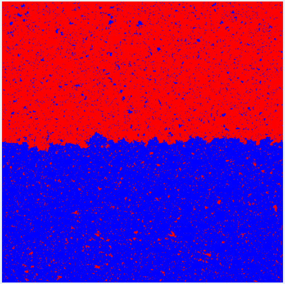
$\Downarrow$ **Why switching fails**

No available switching lemma turns $\mathrm{Var}(\phi_0)$ into a connectivity problem!

$\mathrm{Var}(\phi_0)$ creates a **degree defect** at 0 (not a two-point source $\partial\mathbf{n} = \{0, x\}$).

Random Surfaces and the DGC

2D Ising Interface



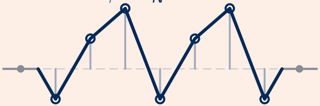
Dobrushin boundary condition

Credit: V. Beffara

?

Discrete Gaussian Chain

$\phi : \Lambda_N \to \mathbb{Z}$



Heights $\phi_i \in \mathbb{Z}$

Conclusion & Perspectives

Summary: 1D Long-Range Ising & Random Currents

- **1D Long-Range Ising Model (DCGT 2025):**

$$\text{Bernoulli} \longrightarrow \text{FK } (q=2) \longrightarrow \text{1D LR Ising}$$

- **Random Current Representation (ADCS 2015):**

$$\langle \sigma_i \sigma_j \rangle^2_{\Lambda,\beta_I} = \mathbb{P}^{\varnothing}_{\Lambda,\beta_I} \otimes \mathbb{P}^{\varnothing}_{\Lambda,\beta_I} \big(i \leftrightarrow j \text{ in } \operatorname{supp}(\mathbf{n}_1 + \mathbf{n}_2) \big)$$

Perspectives

- **Garban's Open Problem 4**
- **Random Current Representation for the DGC**
- **Random Interface $\longleftrightarrow$ DGC**

Thank you for your attention!

Questions & Comments

Master's Defense — September 2, 2026
Sorbonne Université