Nets in Banach Space

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Definition (Separated net)

If $a, b > 0$, an (a, b)-*net* or (a, b)-separated net in a normed space $(X, \|\cdot\|)$ is a subset $\mathcal{N} \subset X$ which is:

- *a-separated*, i.e. for every $u \neq v \in \mathcal{N}$, we have $\|u - v\| \geq a$;
- *b-dense*, i.e. for every $x \in X$ there exists $u \in \mathcal{N}$ such that $\|x - u\| \leq b$.

For example, the grid $\mathbb{Z}^2 \subset \mathbb{R}^n$ is a $(1, \frac{\sqrt{n}}{2})$-net.

Definition

Definition (Bi-Lipschitz mapping)

A map $\phi : \mathbb{R}^n \to \mathbb{R}^n$ is *bi-Lipschitz* if there is a constant K such that

$$\frac{1}{K} < \frac{\|\phi(x) - \phi(x')\|}{\|x - x'\|} < K$$

for $x \neq x'$, where $\| \cdot \|$ is the Euclidean norm.

Definition

Definition (Bi-Lipschitz equivalent subsets)

Two subsets $\mathcal{N}_1$, $\mathcal{N}_2$ of a normed space $(X, \|\cdot\|)$ are bi-Lipschitz equivalent if there exists a bi-Lipschitz and bijective map from $\mathcal{N}_1$ to $\mathcal{N}_2$.

Gromov's question

Is it true that any two nets in the same Euclidean space $\mathbb{R}^n$ are **bi-Lipschitz equivalent**, for $n \geq 2$?

A Negative Answer:

- A counter-example found independently by Burago–Kleiner and McMullen in 1998.

Follow-up Question

What about in the infinite-dimensional Banach space?

A Positive Answer:

- Shown by Lindenstrauss, Matoušková, and Preiss in 2000

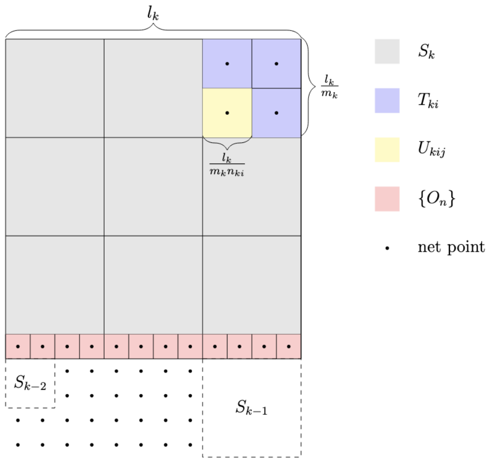
Burago-Kleiner, McMullen 1998

Theorem 1 (Burago-Kleiner, McMullen 1998)

There exists an $(a, b)-$net in $\mathbb{R}^2$ which is not bi-Lipschitz equivalent to $\mathbb{Z}^2$.

Construction of a separated net (Sketch for $k = 3$)

- Define an affine map $\phi_k : [0,1]^2 \to S_k$, and set $\rho_k = \frac{1}{\rho \circ \phi_k^{-1}}$ on S_k.
- $n_{ki} = \lfloor \sqrt{\int_{T_{ki}} \rho_k \, d\mathcal{L}} \rfloor$.



Equivalent Theorem

Equivalent Theorem

Let $\rho : [0,1]^2 \to [1, 1+c]$ be a measurable function which is not the Jacobian of any bi-Lipschitz map $\varphi : [0,1]^2 \to \mathbb{R}^2$ with

$$\mathrm{Jac}(f) := \det(Df) = \rho \quad \text{a.e.}$$

If and only if there exists an (a, b)-net in $\mathbb{R}^2$ which is not bi-Lipschitz equivalent to $\mathbb{Z}^2$.

Sketch of Proof : $\Longrightarrow$

Suppose exist g bi-lipschitz homeomorphism from $\mathcal{N}$ to $\mathbb{Z}^2$

- Define rescaled maps f_k from $\mathcal{N}_k := \mathcal{N} \cap S_k \subset [0,1]^2$ to $\mathbb{R}^2$.

$$f_k(x) = \frac{1}{l_k}\left(g \circ \phi_k(x) - g \circ \phi_k(\star_k)\right)$$

  where $\star_k$ is some point $\in \mathcal{N}_k$
- By proof of Arzelà-Ascoli (diagonal argument), extract a uniformly convergent subsequence $f_k \to f$.
- Measure pushforward implies: $\mathrm{Jac}(f) = \rho$ a.e.

Sketch of Proof : $\Longleftarrow$

- Shown by McMullen

Counter-example

Lemma

For every $\lambda > 1$ and $L > 1$, there exists $\mu > 1$ with the following property:
Let $x, y \in \mathbb{R}^2$, and let U be an open neighborhood of the segment $[x, y]$. Then there exist a function f from U to $\{1, \lambda\}$, a finite set of disjoint segments $I_j = [x_j, y_j] \subset U$ and $\varepsilon > 0$, such that if $\varphi : U \to \mathbb{R}^2$ is a Lipschitz homeomorphism with Lipschitz constants of φ and φ^{-1} bounded by L, and

$$\mathcal{L}\left(\{u \in U : Jac(\varphi)(u) \neq f(u)\}\right) < \varepsilon,$$

then there exists a j such that

$$\frac{|\varphi(x_j) - \varphi(y_j)|}{\|x_j - y_j\|} \geq \mu \cdot \frac{|\varphi(x) - \varphi(y)|}{\|x - y\|}.$$

- Define $f = 1$ on even-indexed S_i, and $f = \lambda$ on odd-indexed S_i.
- for each $1 \leq i \leq N$,$0 \leq k < M$ and $0 \leq l \leq M$

$$z_{k,l}(i) = \left(\frac{i-1}{N} + \frac{k}{MN}, \frac{l}{MN} \right)$$

- $\varphi : U \to \mathbb{R}^2$ be a L-bi-Lipschitz homeomorphism :
  - $\varphi(x) = (0,0)$ and $\varphi(y) = (a,0)$ where $x = (0,0)$ and $y = (1,0)$

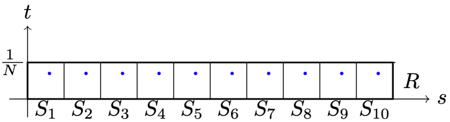


Figure: Grid structure in R for $N = 10$; blue points denote $z_{k,l}(i)$

- $w_{k,l}(i) = \varphi(z_{k,l}(i)) - \varphi(z_k(i+1))$
- $w = (a/N, 0)$
- there exist i, for each k and l, $\|w_{k,l}(i) - w\| < \eta a/N$

Definition (Haussdorff distance)

The Hausdorff distance between X and Y is defined as

$$d_H(X, Y) := \max\left\{ \sup_{x\in X} d(x, Y), \sup_{y\in Y} d(X, y) \right\},$$

where

$$d(a, B) := \inf_{b\in B} d(a, b)$$

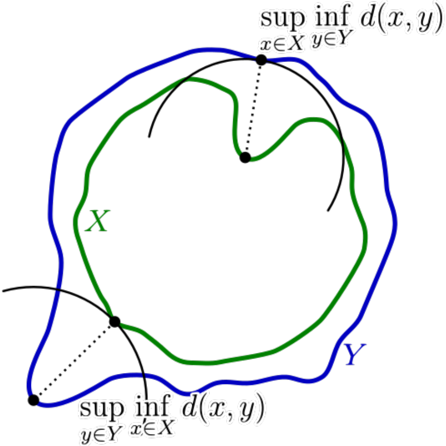


Figure: Components of the calculation of the Hausdorff distance between the green curve X and the blue curve Y

- $d_H(\varphi(S_i) + w, \varphi(S_{i+1})) \leq \eta a/N + L/(MN)$
- By Area of σ-neighborhood bounded:

$$|\mathcal{L}(\varphi(S_i) + w) - \mathcal{L}(\varphi(S_{i+1}) + w)| < \frac{\lambda - 1}{2N^2}$$

- By definition of Area:

$$|\mathcal{L}(\varphi(S_i) + w) - \mathcal{L}(\varphi(S_{i+1}) + w)| \geq \frac{\lambda - 1}{N^2} - 2\varepsilon L^2$$

- Contradiction of $\varepsilon \implies$ lemma

- Step 1:
  - Apply lemma : exist g_1 on U_L, $\{I_{1,j}\}$ and ε_1
  - Choose $U_{1,j}$ containing $I_{1,j}$ such that $\mathcal{L}(\cup_j U_{1,j}) < \varepsilon_1/2$
- Step 2:
  - Apply lemma: exist $g_{1,j}$ on $U_{1,j}$, $\{I_{1,j,k}\}$ and $\varepsilon_{1,j}$
  - $$g_2 = \begin{cases} g_1 & on\ U_L \setminus \cup_j U_{1,j} \\ g_{1,j} & on\ U_{1,j} \end{cases}$$
  - Choose $U_{1,j,k}$ containing $I_{1,j,k}$ such that $\mathcal{L}(\cup_{j,k} U_{1,j,k}) < \min_j \varepsilon_{1,j}/2$

- Suppose $\varphi : U_L \to \mathbb{R}^2$ is bi-Lipschitz with $J(\varphi) = f_L$ a.e.
- Then in each iteration, φ must stretch at least one segment $[u, v]$ by a factor $\geq \mu^n / L$.
- After k steps, this accumulates to a stretch $\geq \mu^k / L > L$, contradicting the L-Lipschitz condition.

Conclusion: No such bi-Lipschitz φ exists.

Theorem 2 (Lindenstrauss, Matoušková, Preiss 2000)

Let E be an infinite-dimensional Banach space. Then any two nets in E are bi-lipschitz equivalent.

- Any net in E has the same cardinality equal to density character of X.

Lemma

Let X be a metric space, and let $\mathcal{N}_1$ and $\mathcal{N}_2$ be two nets in X. Let $T : \mathcal{N}_1 \to \mathcal{N}_2$ be a bijective map, such that $\sup\{d(x, Tx) : x \in \mathcal{N}_1\} < \infty$. Then T is a bi-Lipschitz equivalence between the two nets.

- Choose a **maximal 5b-separated** set of points $\{y_\alpha\}$, and use it to partition the space E into a series of "small blocks" C_α with $C_\alpha = \{x \in$E: $\|x - y_\alpha\| \leq 5b\}$ where $diam(C_\alpha) \leq 10b$
- $Card(C_\alpha \cap \mathcal{N}_1) = Card(C_\alpha \cap \mathcal{N}_2)^*$

* the equality is not true in finite-dimensional space

Conclusion

- Burago-Kleiner, McMullen 1998
- Lindenstrauss, Matoušková, Preiss 2000

Thank you for your attention!

Question?